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# Passage Time Statistics in a Stochastic Verhulst Model

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**Abstract** Using the stochastic paths perturbation approach analytic individual realizations of a stochastic Verhulst model are introduced. The escape of the unstable state is studied for any kind of noise from these individual realizations. We infer from these paths the statistics of the first passage time distribution invoking the solution of an explicit equation with a random coefficient. A stochastic population Verhulst's dynamics with small perturbations of the Wiener class is explicitly worked out. The method can also be implemented for other type of stochastic perturbations like Poisson-noise (shot white pulses), etc.

**Keywords** Population dynamics · Stochastic Verhulst model · First passage time

## 1 Introduction

### 1.1 The Stochastic Path Perturbation Approach

Nonlinear systems far from equilibrium exhibit a variety of instabilities when the appropriate control parameters are changed [1]. By such changes of the control parameters the system can be placed in an unstable state [2]. The system, in general, will relax to a metastable (or global) stationary state [3]. This transient process is triggered by any noise of amplitude $\epsilon$, while the statistical description of such a transient constitutes one of the main subjects of nonequilibrium statistical mechanics [4–6]. A detailed description of the relaxation process

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depends on the nature of the instability involved [7–9]. Unstable states appear in first-order-like instabilities at the end point of hysteresis cycles. Typical cases are those possessing in the order parameter the symmetry transformation $n \to -n$ in the relaxation from $n = 0$. In particular, when there is such an inversion symmetry, a theory for the relaxation at a subcritical pitchfork bifurcation has recently been introduced quite successfully [7]. That approach is based on the fact that each stochastic path (up to $\mathcal{O}(\sqrt{\epsilon})$) can be approximated systematically with a suitable perturbation on the deterministic one. Therefore the lifetime from an unstable state, can be studied in terms of random escape times $t_e$, which in fact, are governed by those approximated stochastic paths (quasideterministic paths [4]). That theory allows us, in principle, to find the *lifetime* of any unstable state (i.e.: the passage time to some macroscopic value $n \approx \mathcal{O}(1)$) [10, 11]. The lack of an initial Gaussian regime does not pose any restriction for determining the statistical properties of the lifetime from an unstable state [7].

Here we use this Stochastic Path Perturbation Approach (SPPA) for tackling the problem of the passage times statistics in a population dynamics model, and we specially focus on the analytic expression for the First Passage Time Distribution (FPTD). Thus all the moments of the passage time can easily be calculated. We point out that in general the time-scale characterizing the escape from the instability, is the lifetime of the state calculated as the Mean First Passage Time (MFPT). Also the study of the transient relaxation of the system, i.e.: the anomalous fluctuations of the phase space variable (the moments of the order parameter) can analytically be calculated introducing an instanton-like approximation [7, 12], this will be done in Sect. 2.5.

When the perturbation on some deterministic dynamics is a Gaussian white noise, the standard theory of continuous stochastic processes gives the FPTD by solving the corresponding adjoint Fokker-Planck operator [13–15]. The problem presented here is the characterization of the time-scale of the escape process by looking at each stochastic realization of the stochastic dynamics. In this way we are going to define a random escape time, $t_e$, as the random time when the amplitude (population size) $n(t)$ reaches a given threshold. This means that the escape time $t_e = t_e(\Omega)$ is going to be a function of a random number, $\Omega$, which will be correctly characterized by a certain probability measure $P(\Omega)$. Then, in principle, all the moments of $t_e$ can be calculated by taking the mean value over $P(\Omega)$. This picture has the advantage over the usual Fokker-Planck technique because it displays the existence of the relevant physical universal parameter of the system in a very direct way. On the other hand, the SPPA picture allows the *analytic* calculation of the FPTD $P(t_e)$ as a perturbation in the small noise intensity $\sqrt{\epsilon}$.

The FPTD is obtained from the use of the transformation of the random variable theorem, and it will be shown to be a non-symmetric broad distribution peaked around the most probable value. Verhulst's population model with small perturbations of the Wiener class is explicitly worked out. The present method can also be implemented for other type of stochastic perturbations like Poisson-noise (shot white pulses), Dichotomic-noise [14, 15], etc. this is pointed out Sect. 3.

## 2 Application to Population Dynamics: the Logistic Model

Biological problems in which fluctuations are important to be consider are for example in models of population dynamics. In order to apply our SPPA to study the time-scales in a population dynamics system, we consider here the Verhulst model

$$\frac{dn}{dt} = r\left(1 - \frac{n}{K}\right)n, \qquad n(t) > 0, \quad \forall t \geq 0, \tag{1}$$

where $K$ is the maximum population that a given amount of food can support, and $r$ the net growth rate per individual [13]. Equation (1) can be integrated and its solution is the so-called logistic curve which has a characteristic S-like shape. In this deterministic case the typical time-scale is $t_d \sim 1/r$, nevertheless a more realistic model is needed to represent the fact that in a real experiment the characteristic time-scale, i.e., the instant when the population size $n(t)$ is of macroscopic order $\mathcal{O}(K)$, is a random time. A more realistic description can be obtained if we generalize the deterministic evolution (1) by a stochastic one adding small fluctuations in the form of a Stochastic Differential Equation (SDE)

$$\frac{dn}{dt} = r\left(1 - \frac{n}{K}\right)n + \sqrt{\epsilon}\,\xi(t), \tag{2}$$

here $\sqrt{\epsilon}$ characterizes the size of the fluctuations of any arbitrary noise $\xi(t)$, which must be a *bounded* process in such a way to assure that each realization of the stochastic process $n(t)$ fulfills $n(t) > 0,\ \forall t \geq 0$.

Following the SPPA we introduce for the present model the transformation

$$n = \frac{Z}{Y^\theta}, \quad \text{then} \quad \dot{n} = \frac{\dot{Z}}{Y^\theta} - \theta \frac{Z\dot{Y}}{Y^{\theta+1}}.$$

From this it is simple to see that we can chose two SDE of the form

$$\begin{aligned} \dot{Z} &= rZ + \sqrt{\epsilon} Y^\theta \xi, \\ -\theta \frac{Z\dot{Y}}{Y} &= -\frac{r}{K}\frac{Z^2}{Y^\theta}. \end{aligned}$$

Taking $\theta = 1$ we get two SDE that can easily be iterated

$$\dot{Z} = rZ + \sqrt{\epsilon} Y \xi, \tag{3}$$

$$\dot{Y} = \frac{r}{K} Z. \tag{4}$$

## 2.1 The Deterministic Case

The deterministic solution can be reobtained from (3) and (4) taking $\sqrt{\epsilon} = 0$. In this case we get

$$\begin{aligned} Z(t) &= Z_0 e^{rt}, \\ Y(t) - Y_0 &= \frac{rZ_0}{K}\int_0^t e^{rs} ds = \frac{rZ_0}{K}\left(\frac{e^{rt} - 1}{r}\right). \end{aligned}$$

Then we arrive to the well known logistic S-like shape

$$n(t) = \frac{Z(t)}{Y(t)} = \frac{K}{(\frac{K}{n_0} - 1)\exp(-rt) + 1}, \quad n_0 \equiv \frac{Z_0}{Y_0},\ K > n_0,\ t \geq 0. \tag{5}$$

This solution shows that at the deterministic time $t_d = \frac{1}{r}\ln(\frac{K}{n_0} - 1)$, when $\dot{n}(t)$ is maximum, the population size is $n(t_d) = K/2$.

### 2.2 The Stochastic Paths to $\mathcal{O}(\sqrt{\epsilon})$

These paths can be obtained from (3) and (4) iterating two times. From now on we will be interested only in the transition $\mathcal{O}(\sqrt{\epsilon})$ to $\mathcal{O}(K)$ in the phase space variable $n(t)$. Therefore consider the initial conditions $Z(0)=0$ and $Y(0)=1$, then from (3) we get, to $\mathcal{O}(\sqrt{\epsilon})$

$$Z(t) \simeq \sqrt{\epsilon} e^{rt} \int_0^t e^{-rs} \xi(s) ds \equiv \sqrt{\epsilon} e^{rt} h(t), \quad t \geq 0, \tag{6}$$

where we have defined a new stochastic process $h(t) \equiv \int_0^t e^{-rs}\xi(s)ds \geq 0$, which is solution of the SDE:

$$\dot{h}(t) = e^{-rt}\xi(t), \qquad h(0)=0, \quad t \geq 0. \tag{7}$$

Using the short-time solution (6) in (4) and iterating for $Y(t)$ we get, to $\mathcal{O}(\sqrt{\epsilon})$

$$Y(t) - Y(0) \simeq \frac{r}{K} \int_0^t Z(s) ds = \frac{r\sqrt{\epsilon}}{K} \int_0^t e^{rs} h(s) ds,$$

which means that we can write for the stochastic path $n(t) = Z(t)/Y(t)$ the representation

$$n(t) \simeq \frac{\sqrt{\epsilon} h(t) \exp(rt)}{1 + \frac{r\sqrt{\epsilon}}{K} \int_0^t e^{rs} h(s) ds}, \quad t \geq 0. \tag{8}$$

It is important to note that even when the correlation function of the arbitrary process $\xi(t)$ could be white, the stochastic process $h(t)$ saturates at long time ($t \gg r^{-1}$), then we can approximate the paths (8) in the form:

$$n(t) \simeq \frac{\sqrt{\epsilon} h(\infty) \exp(rt)}{1 + \frac{r\sqrt{\epsilon}}{K} h(\infty) \int_0^t e^{rs} ds} = \frac{K}{(\frac{K}{\sqrt{\epsilon}h(\infty)} - 1)\exp(-rt) + 1}, \quad t \geq 0. \tag{9}$$

Formula (9) gives the stochastic paths we were looking for as a mapping from the random number $h(\infty)$. This solution gives an accurate representation of the paths for short and intermediate times, except for the small fluctuations around the final steady state $n(\infty) = K$, i.e., in the long-time limit $t \to \infty$.

At *intermediate* times, the random scape times $t_e$ (when the stochastic paths leave the initial domain $\mathcal{O}(\sqrt{\epsilon})$ and fall into the attractor of the saturation valley) can be obtained by inverting $t$ from (9). To $\mathcal{O}(\sqrt{\epsilon})$ we get

$$\left(\frac{K}{\sqrt{\epsilon}h(\infty)} - 1\right) \exp(-rt_e) \sim 1, \tag{10}$$

then from (10) we can write asymptotically for small noise

$$t_e = \frac{1}{r} \log\left(\frac{K}{\sqrt{\epsilon}\Omega}\right), \quad t_e \geq 0, \tag{11}$$

where we have used the notation $\Omega \equiv h(\infty)$, here we emphasize that this random variable must have a positive support. This formulae teaches us that the MFPT is just given by the mean value $\langle t_e \rangle$ over the distribution of the random variable $\Omega$ (into a suitable support to assure the positivity of $t_e$).

In order to obtain the probability density of the escape times $P(t_e)$ (i.e., the probability that amplitude $n(t)$ reaches a given threshold, $\mathcal{O}(K)$, between $t_e$ and $t_e + dt_e$), we begin with the relation between $\Omega$ and $t_e$ expressed in (11). Assuming that $P(\Omega)$ is known and using the transformation of the random variables theorem [14, 15], it is possible to calculate the FPTD $P(t_e)$ as:

$$P(t_e) = \int \delta\left(t_e - t_e\left[\Omega\right]\right) P(\Omega) d\Omega.$$

Note that in this case the supports of $\Omega$ and $t_e$ do not correspond to each other. Due to the fact that the transformation (11) has a single-value inverse and using that $|\frac{d\Omega}{dt_e}| = r\Omega$, we write

$$P(t_e) = \frac{1}{\mathcal{N}} \frac{rK}{\sqrt{\epsilon}} \exp(-rt_e) P\left(\Omega = \frac{K \exp(-rt_e)}{\sqrt{\epsilon}}\right), \quad t_e \in (0, \infty), \tag{12}$$

where $\mathcal{N}$ is a normalization constant. Therefore our the next task is to characterize the probability measure $P(\Omega)$.

In order to compare the probability density $P(t_e)$ against a numerical simulation of the escape times, we need to specify the threshold for the amplitude $n(t_e)$. We have carried out simulations from the SDE (2) taking $n(t_e) = K/2$ and considering that $\xi(t)$ is a Gaussian white noise, see Appendix A. Comparing with the simulations we have seen that the approximation described above has a small systematic underestimation of the escape times [7, 11]. To solve this issue we introduce a simple *redefinition* of the constant $K$ in our approach. Denoting by $Z_j(t), Y_j(t)$ the $j$th-iteration in the SPPA we know from the first iteration, equation (6), that the escape process is controlled by the linear regime, then at the escape time $t_e$ the size of the amplitude $n(t) = Z(t)/Y(t)$ will be asymptotically, for small noise

$$n(t) \sim \frac{Z_0(t)}{Y_0(t)} \sim \frac{\sqrt{\epsilon} e^{rt} h(t)}{1} \to \sqrt{\epsilon}\Omega \exp(-rt_e) \sim \mathcal{O}(K) = K'. \tag{13}$$

From the second iteration of the SPPA we know that the amplitude $n(t)$ is controlled by the nonlinear deterministic evolution, see (8). Then at the scape time the size of $n(t_e)$ will asymptotically be

$$n(t) \sim \frac{Z_0(t)}{Y_1(t)} \to \frac{K}{\frac{K}{\sqrt{\epsilon}\Omega} \exp(-rt_e) + 1} \sim \frac{K}{2}. \tag{14}$$

Comparing (13) and (14) we get $K' = \frac{1}{2}K$, giving a *renormalized* value for the constant $K$. In this work we have carried out numerical simulations considering Heun's algorithm when the process $\xi(t)$ is a Gaussian white noise (see Appendix A); Fig. 1 shows the result of taking this procedure into account.

### 2.3 The Probability Distribution $P(\Omega)$ for Wiener Perturbations

The FPTD, $P(t_e)$, of the amplitude $n(t)$ can be obtained from the asymptotic statistics of the stochastic process $h(t)$ which is solution of the SDE (7). In order to work out the process $h(t)$ we have to give the statistics of the noise $\xi(t)$. As we emphasize in previous sections our approach can be applied to many different noises, in particular in the present section we are going to specify $\xi(t)$ as a zero mean Gaussian white noise. Nevertheless, even in this case we have to work out the Gaussian case with care because we need to assure that the process $h(t)$ must have positive realizations. This can be done by introducing reflecting boundary

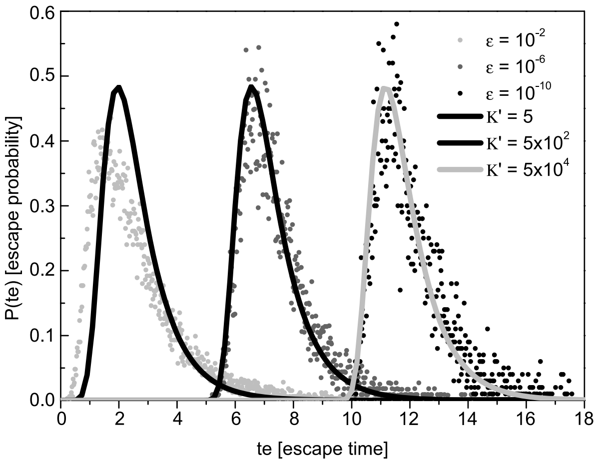


**Fig. 1** Plots of the FPTD given in (24), $P(\tau_e)$, coming from the present SPPA for a Gaussian noise perturbation, as a function of $\tau_e = rt_e$ for $\tilde{K} \equiv K\sqrt{r/\epsilon}$ $(= 10, 10^3, 10^5)$, where we have used the renormalized value $K' = \frac{1}{2}\tilde{K}$. The *dotted curves* represent the Monte Carlo simulations of the SDE (2) with $K = r = 1$, for three different values of the noise amplitude $\epsilon$ $(= 10^{-2}, 10^{-6}, 10^{-10})$, having reached $n(t) = K/2$ for the first time. Details of the numerical simulation are given in Appendix A

condition in Wiener integrals. Note that the method of image can also be used when working with non-Gaussian noises if we know its corresponding characteristic function [16].

Using $\xi(t)dt = dW(t)$ we can write the *unbounded* solution of (7) in the form

$$\int dh_0(t) = \int e^{-rs} dW(s), \tag{15}$$

where $dW(t)$ is a Wiener differential with zero mean value. As we have noted in (6) we are interested in the initial condition $Z(0) = 0$. Using the properties of the Wiener integral we can calculate any moment of the unbounded process $h_0(t)$. In particular, because $h_0(t)$ is Gaussian, to calculate the 1-time probability distribution $P(h_0, t)$ we only need the first and second moment of $h_0(t)$. From (15) we get

$$\begin{aligned} \langle h_0(t)\rangle &= 0, \\ \langle h_0(t)^2\rangle &= \frac{1}{2r}[1 - \exp(-2rt)] \equiv \sigma^2(t), \quad t \geq 0. \end{aligned} \tag{16}$$

Then the 1-time probability distribution of the *unbounded* process $h_0(t)$ is given by

$$P(h_0, t) = \frac{1}{\sqrt{2\pi\sigma^2(t)}} \exp\left(-\frac{h_0^2}{2\sigma^2(t)}\right), \quad h_0 \in (-\infty, \infty),\ t \geq 0. \tag{17}$$

To assure that the stochastic process $h(t)$, appearing in (8), has positive realizations we now use the method of image to built up a stochastic process fulfilling this condition. The constructions of these bounded realizations $h(t)$ follows by the use of a negative mirror image (around the origin) of the positive one. Using that the initial condition is $h(0) = 0$, from a corollary of theorem 2 of Ref. [16] we finally get

$$P(h, t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \cos(kh) \exp\left(-\frac{\sigma^2(t)k^2}{2}\right) dk, \tag{18}$$

then

$$P(h, t) = \frac{\sqrt{2}}{\sqrt{\pi\sigma^2(t)}} \exp\left(-\frac{h^2}{2\sigma^2(t)}\right), \quad h \in (0, \infty),\ t \geq 0, \tag{19}$$

this formula gives $P(\Omega)$ in the limit $P(h, t \to \infty)$.

From this expression we immediately can calculate the generating function of $h(t)$ for a fixed time $t$

$$\begin{aligned} G_h(\lambda) &= \int_0^\infty e^{-h\lambda} P(h,t)dh \\ &= \operatorname{erfc}[\lambda\sigma(t)/\sqrt{2}]\exp(\lambda^2\sigma^2(t)/2). \end{aligned} \tag{20}$$

Then the 1-time moments are given by $\langle h^m(t)\rangle = (-1)^m d^m G_h(\lambda)/d\lambda^m|_{\lambda=0}$, $m = 1, 2, 3\ldots$, for example

$$\langle h(\infty)\rangle = \sqrt{\frac{2}{\pi}}\sigma(t=\infty) = \frac{1}{\sqrt{\pi r}}.$$

In addition from the generating function, $G_h(\lambda)$, any inverse moment of $h(t)$ can be calculated by quadrature in the form

$$\left\langle \frac{1}{h^\nu(t)} \right\rangle = \frac{1}{\Gamma(\nu)} \int_0^\infty G_h(\lambda)\lambda^{\nu-1}d\lambda, \quad \nu > 0, \tag{21}$$

then, for example it is simple to see that $\langle \frac{1}{h^\nu(t)} \rangle \neq \infty$ only for fractional inverse moments with $\nu < 1$.

### 2.4 The FPTD for Verhulst's Model with Wiener Perturbations

Here we are interested in the long-time limit of the process $h(t)$, which allow us to write the statistical properties of the random variable $h(\infty)$. From (19) and $\sigma^2(t=\infty) \equiv 1/2r$ we get

$$P(\Omega) = 2\sqrt{\frac{r}{\pi}}\exp(-r\Omega^2), \quad \Omega \in (0,\infty). \tag{22}$$

Formula (22) completes the description of the FPTD for our logistic dynamics (2) perturbed with *bounded* Gaussian fluctuations of intensity $\sqrt{\epsilon}$. In fact, to the dominant $\mathcal{O}(\sqrt{\epsilon})$, using (12) the distribution of the random scape times $t_e$ is given by

$$P(t_e) = \frac{2K}{\mathcal{N}} \frac{r^{3/2}\exp(-rt_e)}{\sqrt{\pi\epsilon}} \exp\left(-\frac{K^2\exp(-2rt_e)}{\epsilon/r}\right), \quad t_e \in (0,\infty). \tag{23}$$

Note that this formula has only one important universal parameters $\tilde{K} \equiv K\sqrt{r/\epsilon}$, therefore introducing the change of variable $\tau_e \equiv rt_e$ we obtain a universal dimensionless expression for the FPTD

$$P(\tau_e) = \frac{2\tilde{K}}{\operatorname{erf}(\tilde{K})\sqrt{\pi}} \exp[-\tau_e - \tilde{K}^2\exp(-2\tau_e)], \quad \tilde{K} \equiv K\sqrt{r/\epsilon}, \tau_e \equiv rt_e. \tag{24}$$

Figure 1 depicts the $P(\tau_e)$ curves for different values of $\tilde{K}$ $(= 10, 10^3, 10^5)$. Note that these functions are plotted considering the renormalization procedure, see (13) and (14), i.e., using $\frac{1}{2}\tilde{K}$. Also the corresponding Monte Carlo simulations are shown for the same set of parameters, i.e., we use $K = r = 1$ and change the noise amplitude $\epsilon$ $(= 10^{-2}, 10^{-6}, 10^{-10})$. The agreement is very good even for the case of large noise (which corresponds to small $\tilde{K}$).

In particular from (24) the *lifetime* of our perturbed Verhulst's model, i.e., the MFPT for the stochastic logistic dynamics, is asymptotically for small noise given by

$$\langle \tau_e \rangle = \int_0^\infty \tau_e P(\tau_e) d\tau_e \simeq \ln\left(\frac{1}{2}\tilde{K}\right) + \frac{\mathcal{E} + \ln 4}{2\,\mathrm{erf}(\frac{1}{2}\tilde{K})}, \quad \tilde{K} \gg 1, \tag{25}$$

here $\mathcal{E} = 0.577215\ldots$ is the Euler constant and we have introduced the renormalized value $\frac{1}{2}\tilde{K}$. This formula gives agreement, even in the case of small $\tilde{K}$, when compared with the numerical simulations of the *lifetime*. For the particular set of parameters $\{K = 1, r = 1\}$ used in figure 1, we made the following table:

| | $\epsilon = 10^{-2}$ | $\epsilon = 10^{-6}$ | $\epsilon = 10^{-10}$ |
|---|---|---|---|
| $\langle \tau_e \rangle$ | 2.59 | 7.19 | 11.80 |
| $\langle \tau_e \rangle_{MC}$ | 2.52 | 7.35 | 11.97 |

These values show the good accuracy of formula (25) with the lifetime evaluated from the Monte Carlo simulations. In addition to this check we have performed a *goodness of fit test* for our theoretical distribution function $P(\tau_e)$, this can be seen in Appendix B.

### 2.5 Transient Fluctuations with Wiener Perturbations

From the approximate paths (9), defining $N(t) = n(t)/K$ we can write to $\mathcal{O}(\sqrt{\epsilon})$

$$N(t) = \frac{1}{(\frac{K}{\sqrt{\epsilon}\Omega} - 1)\exp(-rt) + 1}, \quad t \geq 0. \tag{26}$$

Introducing the dimensionless units of time $\tau = rt$ the mean value of the population size can be calculated using the distribution $P(\Omega)$, from (22) we get

$$\begin{aligned}
\langle N(\tau)\rangle &= 2\sqrt{\frac{r}{\pi}} \int_0^\infty \frac{\exp(-r\Omega^2)d\Omega}{(\frac{K}{\sqrt{\epsilon}\Omega} - 1)e^{-\tau} + 1}, \quad \tau \geq 0 \\
&= \frac{2e^\tau}{\sqrt{\pi}\tilde{K}} \int_0^\infty \frac{x\exp(-x^2)dx}{1 + \alpha x/\tilde{K}}, \quad \alpha \equiv e^\tau - 1,\ \tilde{K} \equiv K\sqrt{r/\epsilon}.
\end{aligned}$$

Then for any time $\tau$ (except in the limit $\tau \to \infty$) this expression gives an asymptotic accurate series representation for the growth of the mean population size

$$\begin{aligned}
\langle N(\tau)\rangle &= \frac{2e^\tau}{\sqrt{\pi}\tilde{K}} \sum_{n=0}^\infty \left(\frac{-\alpha}{\tilde{K}}\right)^n \int_0^\infty x^{n+1}e^{-x^2}dx, \quad \frac{\alpha}{\tilde{K}} < 1 \\
&= \frac{e^\tau}{\sqrt{\pi}\tilde{K}} \sum_{n=0}^\infty \left[\left(\frac{1-e^\tau}{\tilde{K}}\right)^{2n} n! + \sqrt{\pi}\left(\frac{1-e^\tau}{\tilde{K}}\right)^{2n+1} \frac{(2n+1)!!}{2^{m+1}}\right].
\end{aligned}$$

In general the anomalous transient fluctuation in the phase space variable is defined as the mean quadratic deviation of the $N(t)$ process [2]:

$$\sigma_N^2(\tau) = \langle N^2(\tau)\rangle - \langle N(\tau)\rangle^2. \tag{27}$$

To calculate *analytically* the anomalous fluctuation, we approximate the transient toward the global attracting solution by introducing a crude instanton-like approximation [7, 12]

$$N(\tau) = \Theta(\tau - t_e), \quad \text{the Heaviside step function,} \tag{28}$$

which is for $\tau > \tau_e$ the $\mathcal{O}(1)$ macroscopic amplitude of the population size (characterizing the attractor valley). Then the transient anomalous fluctuation is given by

$$\sigma_N^2(\tau) = \langle \Theta(\tau - \tau_e) \rangle - \langle \Theta(\tau - \tau_e) \rangle^2, \quad \tau \geq 0, \tag{29}$$

where

$$\begin{aligned} \langle \Theta(\tau - \tau_e) \rangle &= \int_0^\infty \Theta(\tau - \tau_e)\, P(\tau_e)\, d\tau_e \\ &= \int_0^\tau P(\tau_e)\, d\tau_e \\ &= 1 - \frac{\operatorname{erf}(\frac{1}{2}\tilde{K}e^{-\tau})}{\operatorname{erf}(\frac{1}{2}\tilde{K})}, \quad \tau \geq 0, \end{aligned} \tag{30}$$

here (24) and the renormalized value $\frac{1}{2}\tilde{K}$ has been used.

In this instanton-like approximation the maximum of the function $\sigma_N^2(\tau)$ is at the most probable escape value $\tau_m = \ln \frac{1}{2}\tilde{K} \sim \langle \tau_e \rangle$. The function $\sigma_N^2(\tau)$ depicts for different values of $\tilde{K}$ the qualitative behavior of the anomalous fluctuation we were looking for the present stochastic Verhulst model (perturbed with Gaussian fluctuations). From the behavior of $\sigma_N^2(\tau)$ we see that in the transient regime the initial fluctuations are amplified and gives rise to the transient anomalous fluctuations of $\mathcal{O}(1)$ as compared with the initial or final fluctuations of $\mathcal{O}(\sqrt{\epsilon})$.

## 3 On the Verhulst Model in Presence of Non-Gaussian Noise

As we emphasized in previous sections the SPPA can be applied to different noise perturbations. In fact the problem of calculating the FPTD was reduced to find the probability distribution $P(\Omega)$ associated to the long-time limit of the process $h(t)$, which is characterized by (7). For any arbitrary noise, two important hypothesis were used in order to arrive to this conclusion: first *the small noise* intensity $\mathcal{O}(\sqrt{\epsilon})$, second *the saturation* of $h(t)$ for times $t \gg r^{-1}$. Along the paper we have also emphasized that realizations of the process $h(t)$ must be positive at all time, this fact can be achieved, for arbitrary noise $\xi(t)$, by using the method of the image if the noise $\xi(t)$ is unbounded; see (18) for the case when working with a Gaussian noise. In the case when the noise $\xi(t)$ itself has positive realizations the procedure is simpler.

Here we are going to introduce some general remarks in order to tackle the case when the noise $\xi(t)$ is arbitrary with positive realizations. As we mention before the problem is reduced to find the long-time probability distribution of the SDE (7).

In [21] we have proved that knowing the generating functional of the noise $G_\xi([k(t)])$, the complete characterization of a process $h(t)$, solution of the SDE

$$\dot{h}(t) = e^{-rt}\xi(t), \qquad h(0) \geq 0, \quad t \geq 0,$$

is given by the functional

$$G_h([V(t)]) = e^{ik_0 h(0)} G_\xi\left(\left[e^{-rt}\int_t^\infty V(s)ds\right]\right), \tag{31}$$

where

$$k_0 = \int_0^\infty V(s)ds.$$

Therefore, the 1-time characteristic function of the process $h(t)$, denoted by $G_h(k,t) = \langle \exp ikh(t)\rangle$, follows from evaluating the functional (31) with the test function $V(s) = k\delta(s-t)$. From the 1-time characteristic function the probability distribution $P(h,t)$ can be obtained by Fourier inversion

$$P(h,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk \exp(-ikh) G_h(k,t).$$

Thus the probability distribution that we are looking for is just $P(\Omega) = P(h,\infty)$.

In order to exemplify this general method we consider here the case when $\xi(t)$ is a white Poisson noise, then its functional is [14, 15, 21]

$$G_\xi([k(t)]) = \exp\left[\rho\int_0^\infty \left\{\exp\left[iAk(t')\right] - 1\right\} dt'\right],$$

where $\rho$ represents the average number of events (pulses) per unit of time, and $A$ is the amplitude of the Dirac pulses. Using (31) the functional of the process $h(t)$ will be (considering $h(0) = 0$)

$$G_h\left([V(t)]\right) = \exp\left[\rho\int_0^\infty \left\{\exp\left[iA\exp(-rt')\int_{t'}^\infty V(s)ds\right] - 1\right\} dt'\right]. \tag{32}$$

The 1-time characteristic function of the process $h(t)$ driven by a Poisson noise is obtained by using the test function $V(s) = k\delta(s-t)$ in (32)

$$G_h(k,t) = \exp\left[\rho\int_0^t \left\{\exp\left[iAk\exp(-rt')\right] - 1\right\} dt'\right].$$

From this characteristic function all the 1-time moments $\langle h(t)^n\rangle$ and cumulants $\langle\langle h(t)^n\rangle\rangle$ of the process $h(t)$ can be evaluated. For example, noting that

$$\log G_h(k,t) = \sum_{n=1}^\infty \frac{(ik)^n}{n!}\rho A^n \frac{(1-e^{-nrt})}{nr} \equiv \sum_{n=1}^\infty \frac{(ik)^n}{n!}\langle\langle h(t)^n\rangle\rangle,$$

we immediately prove that hypothesis 2 is fulfilled because:

$$\sigma_h^2(t) = \langle\langle h(t)^2\rangle\rangle = \rho A^2 \frac{(1-e^{-2rt})}{2r}.$$

The 1-time probability distribution of process $h(t)$ is expressed by quadrature as

$$P(h,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk \exp(-ikh) \exp\left[\rho\int_0^t \left\{\exp\left[iAk\exp(-rt')\right] - 1\right\} dt'\right].$$

So the long-time distribution $P(h, t \to \infty)$ gives for the probability distribution $P(\Omega)$ the expression

$$P(\Omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk \exp(-ik\Omega) \exp\left[\rho \int_0^{\infty} \left\{\exp\left[iAk\exp(-rt')\right] - 1\right\} dt'\right] \quad (33)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk \exp(-ik\Omega) \exp\left[\frac{\rho}{r}\{\mathrm{Ei}(iAk) - \ln(A|k|) - \mathcal{E}\}\right], \quad (34)$$

where $\mathcal{E}$ is the Euler constant and $\mathrm{Ei}(x)$ is the exponential integral function [17]. Unfortunately this Fourier transform cannot be done analytically. Having $P(\Omega)$ we can characterize the FPTD of Verhulst's model perturbed with Poisson noise by using (34) in (12).

It should noted that only in the large scale limit $\Omega \to \infty$, the non-analytic structure appearing in (34) cancels out (considering the limit $k \to 0$ into the integral), leading therefore to a Gaussian asymptotic behavior in the limit $\Omega \gg 1$. Nevertheless the FPTD, see (12), is not only controlled by this asymptotic limit, that is the reason why the study of a population dynamics perturbed by non-Gaussian noises can be of great interest in applied biophysics.

## 4 Summary and Conclusions

This paper is inspired in a method developed and already successfully applied to study relaxation from unstable and marginal states [7, 9, 11, 12, 18], and its generalized extended normal form [19]. The first passage time distribution was obtained analytically from the stochastic path perturbation approach. As we have noted this theory can in principle be applied to many similar normal forms perturbed with arbitrary noises. The only restriction is that the approximation is valid as a perturbation in the small intensity of the noise $\sqrt{\epsilon}$, thus allowing to find the characteristic "lifetimes" of the unstable state.

In (25) we give a formula for the *lifetime* in the case when the logistic model is perturbed by Gaussian fluctuations. In the case when the noise perturbation is Gaussian an interesting result found from the structure of the first passage time distribution $P(t_e)$, is the scale invariance property of the group $\tilde{K}$. We have studied the transient anomalous fluctuation that characterizes the transition from $\mathcal{O}(\sqrt{\epsilon})$ to $\mathcal{O}(K)$ in the order parameter $n(t)$, this universal phenomenon occurs when a normal form has a saturation in its potential, i.e., there exist a relaxation toward an attracting solution [2]. We have given a qualitative description for the anomalous behavior of the variance $\sigma_N^2(t) = \langle N(t)^2\rangle - \langle N(t)\rangle^2$ by introducing an instanton-like approximation for the stochastic realization in the phase space, see (29).

We have explicitly found the first passage time distribution to leave the unstable state $n(0) = 0$ of Verhulst's dynamics perturbed by a positive bounded Wiener process, and compare our theoretical predictions against Monte Carlo simulations. Other stochastic perturbations like Poisson-noise [20, 21], Dichotomic-noise, etc. can in principle be worked out in a similar way, see our comment after (33).

Among interesting phenomena to be studied are the population dynamics in non-homogeneous phase space. In this situation we have to take into account the spatial dependence in the order parameter $n(x, t)$, which would appear in the definition of the stochastic evolution (2), for example, by introducing a Laplace operator to represent migration [13]. The stochastic path perturbation approach can also be implemented to tackle this interesting problem [19]. Using previous experience, on extended systems, work along this line is in progress.

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## Appendix A: Monte Carlo Simulations

In the present paper we have accomplished Monte Carlo simulations of the stochastic logistic equation for the case when the process $\xi(t)dt = dW(t)$ is a Wiener differential. When the noise perturbation is Gaussian we can use the Heun algorithm [22] that discretizes the SDE (2):

$$n(t_{i+1}) = n(t_i) + \frac{h}{2}\left[r\left(1 - \frac{n(t_i)}{K}\right)n(t_i) + r\left(1 - \frac{\hat{n}(t_{i+1})}{K}\right)\hat{n}(t_{i+1})\right] + \sqrt{\epsilon h}W_i,$$

with the predicator step

$$\hat{n}(t_{i+1}) = n(t_i) + h\left(r\left(1 - \frac{n(t_i)}{K}\right)n(t_i)\right) + \sqrt{\epsilon h}W_i.$$

Here, $h$ is the time step $h = t_{i+1} - t_i$, and $W_i$ are independent Gaussian distributed random variables with zero-mean value and variance one. These random numbers are generated using the Box-Muller formula [15, 22]

$$W_i = \sqrt{-2\ln \lambda_{i1}}\cos(2\pi\lambda_{i2}),$$

where $\lambda_{i1}$ and $\lambda_{i2}$ are independent random numbers which are uniformly distributed between 0 and 1.

We recorded the escape time $t_e$ at which $n(t_e) \geq K/2$, the escape threshold for the first time. This procedure is repeated $m$ times to get the histograms. To obtain the results shown in the figure 1 we have used this procedure with $m = 100000$ and $h = 0.01$, taking $K = r = 1$ and for three values of the noise amplitude $\epsilon$.

## Appendix B: The $X^2$ Statistics and the Goodness of the Fit Test

The test for goodness of fit consists of testing the hypothesis that a given set of $M$ observations constitutes values of a random variable with a specified distribution function. The $X^2$ statistics is a useful measure of the discrepancy (goodness of fit) between the actual distribution of a set of data points and the theoretical distribution of a random variable of which the data points allegedly are values. This test is particularly important when dealing with experimental situations confronted with marginally small samples. This is not the case in numerical simulations where the number of stochastic realizations can be very large. Nevertheless, for completeness of our statistical investigation of the numerical data we present here the goodness of the fit test (Null-Hypothesis testing).

Consider $q$ disjoint and exhaustive intervals $(+\infty, a_1) = I_1$, $(a_1, a_2) = I_2$, $(a_2, a_3) = I_3$, etc. and then form the numbers

$$M\int_{I_j} P(\tau_e)d\tau_e = E_j,$$

which can be considered as the expected number of observations which fall in the interval $I_j$. Defining $\phi_j$ to be the actual number of observations which are in the interval $I_j$, the following random variable is formed:

$$X^2 = \sum_{j=1}^{q} \frac{(\phi_j - E_j)^2}{E_j}.$$

It is clear that $X^2$ is an effective measure of the discrepancy between the observed occupancy numbers of the data and the expected occupancy numbers, therefore $X^2$ can be considered as a measure of the goodness of fit. A critical value based on the $X^2$ statistics can be calculated (the analysis, however, is rather involved), and this value corresponds to the desired confidence coefficient. If the computed value of $X^2$ were larger than the critical value, the hypothesis that the observations are values of a random variable with probability density $P(\tau_e)$ would be rejected.

In general under the assumption that $M$ is large, we can use the $\chi^2_\nu(y)$ chi-square distribution with $\nu$ degree of freedom to calculate the critical value in terms of a significance level (confidence coefficient) [23].

Due to the fact that our theoretical distribution $P(\tau_e)$ has no free parameters, we have performed the nonlinear least square test considering $\tilde{K}$ as a fitting parameter in the goodness of the fit problem. Nevertheless, none of the test that we have performed shown a significative variation for the renormalized value of $\tilde{K}$ as was predicted, see after (14). We have computed the value of $X^2$, the sum of squares of difference between data and fit values (SSR), the correlation coefficient (R), and the number of degrees of freedom (DOF), using the OriginPro7.5 software. We perform the nonlinear least square fitting analysis between our numerical simulations (using the physical parameters as in Fig. 1) and the theoretical prediction (24), then we got the following significative table

| | $\epsilon = 10^{-2}$ | $\epsilon = 10^{-6}$ | $\epsilon = 10^{-10}$ |
|---|---|---|---|
| $\frac{X^2}{\text{DOF}}$ | 0.00333 | 0.00067 | 0.00056 |
| R | 0.8550 | 0.9750 | 0.9647 |
| SSR | 4.99 | 1.21 | 1.91 |
| DOF | 1499 | 1799 | 3400 |
| $\frac{\tilde{K}}{2}$ | 5 | 500 | 50000 |

This is in according with our previous expectations concerning the goodness of the fit. Then even for large values of the noise amplitude $\epsilon$ the present values of $X^2$ would allow us to accept the hypothesis as true.

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